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Perfect Splines with Boundary Conditions of Least Norm*

CHEN DIRONG

Institute of Mathematics, Academia Sinica, Beijing, 100080, People's Republic of China

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Let $A=(a_{ij})_{i=1,j=0}^{l,r-1}$ and $B=(b_{ij})_{i=1,j=0}^{m,r-1}$ be matrices of ranks l and m, respectively. Suppose that $\tilde{A}=((-1)^j a_{ij})\in \mathrm{SC}_l$ (sign consistent of order l) and $B\in \mathrm{SC}_m$. Denote by $\mathscr{P}_{r,N}(A, B; \nu_1, ..., \nu_n)$ the set of perfect splines with N knots which have n distinct zeros in (0, 1) with multiplicities $\nu_1, ..., \nu_n$, respectively, and satisfy $A\bar{P}(0)=0$, $B\bar{P}(1)=0$, where $\bar{P}(a)=(p(a), ..., P^{(r-1)}(a))^T$. We show that there is a unique $P^*\in\mathscr{P}_{r,N}(A, B; \nu_1, ..., \nu_n)$ of least uniform norm and that P^* is characterized by the equioscillatory property. This is closely related to the optimal recovery of smooth functions satisfying boundary conditions by using the Hermite data.

1. INTRODUCTION

A perfect spline $P(t)$ of degree r with knots $\{\xi_j\}_1^N\subset(0, 1)$, $\xi_1<\cdots<\xi_N$, has the representation

$$P(t)=\sum_{i=0}^{r-1} a_i t^i+\frac{1}{r!}\left[t^r+2\sum_{j=1}^{N}(-1)^j(t-\xi_j)_+^r\right], \tag{1.1}$$

where $\{a_i\}_{i=0}^{r-1}$ are real constants and, as usual, $t_+=\max\{t, 0\}$. The set of all functions of form (1.1) is denoted by $\mathscr{P}_{r,N}$.

Let $A=(a_{ij})_{i=1,j=0}^{l,r-1}$ and $B=(b_{ij})_{i=1,j=0}^{m,r-1}$ be matrices such that

(i) $0\leqslant l, m\leqslant r$, rank $A=l$, rank $B=m$,

(ii) $\tilde{A}=((-1)^j a_{ij})_{i=1,j=0}^{l,r-1}\in\mathrm{SC}_l$, $B\in\mathrm{SC}_m$,

where $A\in\mathrm{SC}_\mu$ means that all non-zero $\mu\times\mu$ subdeterminants of A maintain the same sign, i.e., there exists $\sigma_A\in\{-1, 1\}$ such that

$$\sigma_A A\begin{pmatrix} i_1, ..., i_\mu \\ j_1, ..., j_\mu\end{pmatrix}\geqslant 0,$$

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for every choice of $i_1, \ldots, i_\mu$ and $j_1, \ldots, j_\mu$, where $A\binom{i_1, \ldots, i_\mu}{j_1, \ldots, j_\mu}$ is the subdeterminant of A composed of $i_1, \ldots, i_\mu$ rows and $j_1, \ldots, j_\mu$ columns, respectively.

Given A and B as above, we define functionals as follows:

$$\begin{aligned} A_i f &= \sum_{j=0}^{r-1} a_{ij} f^{(j)}(0), \qquad i = 1, \ldots, l \\ B_i f &= \sum_{j=0}^{r-1} b_{ij} f^{(j)}(1), \qquad i = 1, \ldots, m. \end{aligned} \tag{1.2}$$

For a given set $\mathscr{F}$ of functions such that $f^{(r-1)}(0)$ and $f^{(r-1)}(1)$ exist for $f \in \mathscr{F}$, we denote by $\mathscr{F}(A, B)$ all functions f in $\mathscr{F}$ with $A_i f = 0$ $(i = 1, \ldots, l)$ and $B_i f = 0$ $(i = 1, \ldots, m)$.

Some problems related to boundary conditions (1.2) have been considered. The problem of existence of interpolating spline s with $A_i s = B_j s = 0$ $(i = 1, \ldots, l, j = 1, \ldots, m)$ is discussed in Ref. [7]. The n-widths of $W_\infty^r(A, B)$ in $C[0, 1]$ are obtained [13], where

$$W_\infty^r = \{ f \mid f^{(r-1)} \text{ abs. cont. on } [0, 1], \; \|f^{(r)}\|_\infty \leqslant 1 \}.$$

Given $\{v_i\}_{i=1}^n$ such that

$$1 \leqslant v_i \leqslant r, \qquad r \leqslant \sum_{i=1}^n v_i = N + r - l - m,$$

we use $\mathscr{P}_{r,N}(A, B; v_1, \ldots, v_n)$ to denote the set of functions in $\mathscr{P}_{r,N}(A, B)$ having n distinct zeros $\{x_i\}_{i=1}^n \subset (0, 1)$ with multiplicities $\{v_i\}_{i=1}^n$, respectively. This paper is devoted to study the extremal problem:

$$\inf\{ \|P\| \mid P \in \mathscr{P}_{r,N}(A, B; v_1, \ldots, v_n)\}, \tag{1.3}$$

where $\|\cdot\| = \|\cdot\|_{C[0,1]}$.

Problem (1.3) has been discussed in [1] for $l = m = 0$ (i.e., no boundary condition) and [9] for (1.2) being quasi-Hermite conditions, respectively. We note that [9] only gives proof for the case $m < N$. As for the case $m = N$ (note that $m \leqslant N$), we see below that the proof is somewhat more complicated.

It is well known that perfect splines are very important in the optimal recovery (see [10, 11]). In fact, as in [5, 10], the intrinsic error of the best scheme approximating $f \in W_\infty^r(A, B)$ in $C[0, 1]$ by using information $\{f^{(j)}(x_i) \mid i = 1, \ldots, n, \; j = 0, \ldots, v_i - 1\}$ is equal to $\|P(\bar{X}, \cdot)\|$, where $P(\bar{X}, \cdot) \in \mathscr{P}_{r,N}(A, B; v_1, \ldots, v_n)$ vanishes at $\bar{X} = \{(x_i, v_i)\}$ (see Lemma 1). Hence, the zeros of $P^* \in \mathscr{P}_{r,N}(A, B; v_1, \ldots, v_n)$ that solves problem (1.3) determine the optimal information. Moreover, spline interpolation at the zeros of P^* is an optimal algorithm (see [5]).

The main results of this paper are as follows.

THEOREM 1. *Let* $\{e_i\}_{i=1}^{n+1}$ *be arbitrary positive numbers. Set* $\sigma_1 = 0$, $\sigma_i = \sum_{k=1}^{i-1} \nu_k$, $k = 2, ..., n+1$. *Then there exists a unique* $P \in \mathscr{P}_{r,N}(A, B; \nu_1, ..., \nu_n)$ *with n distinct zeros* $\{x_i\}_{i=1}^{n}$ *and a positive number R such that*

$$RP(y_i) = (-1)^{\sigma+\sigma_i} e_i, \qquad i = 1, ..., n+1, \tag{1.4}$$

where $\sigma \in \{-1, 1\}$ *fixed and* $\{y_i\}_{i=1}^{n+1}$ *satisfy*

$$0 \leqslant y_1 < x_1 < y_2 < \cdots < x_n < y_{n+1} \leqslant 1$$

with $P'(y_i) = 0$ *whenever* $y_i \in (0, 1)$.

THEOREM 2. *There exists a unique perfect spline* $P^* \in \mathscr{P}_{r,N}(A, B; \nu_1, ..., \nu_n)$, *which solves problem* (1.3). *Moreover,* P^* *is characterized by the equi-oscillatory property; i.e., there exist* $n+1$ *points* $\{y_i\}_{i=1}^{n+1} \subset [0, 1]$ *such that*

$$P^*(y_i) = (-1)^{\sigma+\sigma_i} \|P^*\|, \qquad i = 1, ..., n+1,$$

where σ_i *are given as in Theorem* 1.

2. AUXILIARY LEMMAS

Similar to [2], Li [9] proved the following result by the Hobby–Rice theorem.

LEMMA 1. *Given* $\{x_i\}_{i=1}^{n} \subset (0, 1)$, *there is a function* $P \in \mathscr{P}_{r,N}(A, B; \nu_1, ..., \nu_n)$ *such that P vanishes at* $\bar{X} = \{(x_i, \nu_i)\}_{i=1}^{n}$, *where* (x_i, ν_i) *means that* x_i *is a zero with multiplicity* ν_i.

Let $a = (a_i)_{i=1}^{n} \in \mathbf{R}^n/\{0\}$; $S^+(a)$ denotes the maximal number of sign changes in the sequence $a_1, ..., a_n$ where zero terms are arbitrarily assigned values 1 or -1. For example, $S^+(1, 0, 1) = 2$.

LEMMA 2 [12, p. 163]. *For any* $P \in \mathscr{P}_{r,N}$, *it holds that*

$$Z_r(P, (0, 1)) \leqslant N + r - S^+(((-1)^j P^{(j)}(0))_{j=0}^{r}) - S^+((P^{(j)}(1))_{j=0}^{r}), \tag{2.1}$$

where $Z_r(f, I)$ *is the total number of zeros of f at an interval I counting multiplicities not greater than r.*

We call $P \in \mathscr{P}_{r,N}$ a perfect spline with maximal number of zeros if equality holds in (2.1) for P. For such a perfect spline, its zeros and knots satisfy the so-called interlacing conditions. To be precise, we denote by $\{z_i\}_{i=1}^{s}$

and $\{\xi_i\}_{i=1}^{N}$ arranged in natural increasing order the zeros of P at $(0, 1)$ and knots, respectively, where s is the number of zeros of P at $(0, 1)$ counting multiplicities. Then [5]

$$z_{i-\alpha_0} < \xi_i < z_{i+r-\alpha_0}, \tag{2.2}$$

whenever the subscripts are meaningful, where $\alpha_0 = S^+(((-1)^j P^{(j)}(0))_{j=0}^r)$.

It is known that (see, e.g., [13]) for any $P \in \mathscr{P}_{r,N}(A, B)$, $\alpha_0 \geq l$, $\beta_0 \geq m$. Therefore, any function P in $\mathscr{P}_{r,N}(A, B; v_1, ..., v_n)$ is with maximal number of zeros and $\alpha_0 = l$, $\beta_0 = m$.

LEMMA 3. *If $P \in \mathscr{P}_{r,N}$ is with maximal number of zeros, so is $P' \in \mathscr{P}_{r-1,N}$.*

Proof. Set $\alpha_i = S^+(((-1)^j P^{(j)}(0))_{j=i}^r)$, $\beta_i = S^+((P^{(j)}(1))_{j=i}^r)$, $i = 0, 1$. Let x and y be minimal zero and maximal zero of P in $(0, 1)$, respectively.

Now we should distinguish the cases according to the values of α_i and β_i, $i = 0, 1$. Suppose first that $\alpha_0 - \alpha_1 = 1$ and $\beta_0 - \beta_1 = 0$. Since $P^{(r)}(0) \neq 0$, by [12], $t = 0$ is a left Rolle point of P (the definition of Rolle point can be found in [12, p. 28]). By the extended Rolle theorem [12, p. 29], there exists at least a zero of P' in $(0, x)$. Therefore,

$$\begin{aligned} Z_{r-1}(P', (0, 1)) &\geq Z_{r-1}(P', (0, 1)) + Z_{r-1}(P', [x, y]) \\ &\geq 1 + Z_r(P, [x, y]) - 1 = Z_r(P, (0, 1)) \\ &= N + (r-1) - \alpha_1 - \beta_1. \end{aligned}$$

This together with Lemma 2 implies that $P' \in \mathscr{P}_{r-1,N}$ is with maximal number of zeros. The other cases may be treated similarly. So the proof is complete.

Now we introduce some notations from [7] for later use. Let

$$Z = \{x, 0, ..., r-1 \mid x \in (0, 1)\},$$

$$W = \{0, ..., r-1, \xi \mid \xi \in (0, 1)\},$$

and $K(z, w)$ be a function defined on $Z \times W$ as follows:

$$K(x, i) = u_i(x) = x^i,$$

$$K(x, \xi) = (x - \xi)_+^{r-1},$$

$$K(j, \xi) = \frac{\partial^j}{\partial x^j} K(x, \xi)|_{x=1},$$

$$K(j, i) = u_i^{(j)}(x)|_{x=1}.$$

For given $z_1 < \cdots < z_k$, $w_1 < \cdots < w_p$, the Fredholm matrix based on $K(z, w)$ is

$$K\begin{pmatrix} z_1, ..., z_k \\ w_1, ..., w_p \end{pmatrix} = (K(z_i, w_j))_{i,j=1}^{k,p}.$$

If some z_i and/or w_i coincide, the corresponding columns and/or rows are determined by successive derivatives (see [7] for the details). It is shown in [7] that $K(z, w)$ is total positive, i.e.,

$$\det K\begin{pmatrix} z_1, ..., z_k \\ w_1, ..., w_k \end{pmatrix} \geqslant 0. \tag{2.3}$$

for every choice of $z_1 \leqslant \cdots \leqslant z_k$, $w_1 \leqslant \cdots \leqslant w_k$.

LEMMA 4. *Given any* $\{t_i\}_{i=1}^{\tau}$, $\{y_i\}_{i=1}^{s} \subset (0, 1)$ *and* $\{i_j\}_{j=1}^{k} \subset \{j\}_{j=0}^{r-1}$ $(\tau + l + m = k + 2)$ *each of which is arranged in increasing order, we have* $\sigma_{\bar{A}} \sigma_B \sigma \det \Delta \geqslant 0$, *where* $\sigma = (-1)^{\tau l + l(l-1)/2}$ *and* Δ *is a* $(k+s) \times (k+s)$ *matrix whose* jth *column is*

$$(u_{i_j}(t_1), ..., u_{i_j}(t_\tau), A_1 u_{i_j}, ..., A_l u_{i_j}, B_1 u_{i_j}, ..., B_m u_{i_j})^T$$

for $j = 1, ..., k$ *and* $k + j$th *column is*

$$(K(t_1, y_j), ..., K(t_\tau, y_j), 0, ..., 0, B_1 K(\cdot - y_j), ..., B_m K(\cdot - y_j))^T$$

for $j = 1, ..., s$, *with the same interpretation as in* [7] *if some* t_i *coincide.*

Proof. For ease of notations we assume without loss of generality that $i_j = j - 1$, $j = 1, ..., k$. It is obvious that $A_i u_j = a_{ij} j!$.

Expanding Δ by minors based on the $\tau + 1, ..., \tau + l$th rows (if $k < \tau$, then $\det \Delta = 0$) we get

$$\det \Delta = \sigma \sum_{1 \leqslant j_1 < \cdots < j_r \leqslant k} (-1)^{j_1 + \cdots + j_l} A\begin{pmatrix} 1, ..., l \\ j_1, ..., j_l \end{pmatrix} \prod_{i=1}^{l} (j_i - 1)! \det \Delta_{j_1, ..., j_l},$$

where $\Delta_{j_1, ..., j_l}$ is the submatrix of Δ by eliminating its $\tau + 1, ..., \tau + l$th rows and $j_1, ..., j_l$ columns. Let $\{j_i'\}_{i=1}^{k-l}$ arranged in increasing order denote the complements set of $\{j_i\}_{i=1}^{l}$ to $\{i\}_{i=1}^{k}$. It can be verified that $\Delta_{j_1, ..., j_l} = CD$, where

$$C = \begin{pmatrix} I_\tau \\ B \end{pmatrix}$$

(I_τ stands for the $\tau \times \tau$ identity matrix) and

$$D = K\begin{pmatrix} t_1, ..., t_\tau, 0, ..., r-1 \\ j_j' - 1, ..., j_{k-l}' - 1, y_1, ..., y_s \end{pmatrix}.$$

By the Cauchy–Binet formula (see [6, p. 1]) we have

$$\det \Delta_{j_1, ..., j_l} = \sum_{1 \leqslant k_1 < \cdots < k_{\tau+m} \leqslant \tau + r} C\begin{pmatrix} 1, ..., \tau+m \\ k_1, ..., k_{\tau+m} \end{pmatrix} D\begin{pmatrix} k_1, ..., k_{\tau+m} \\ 1, ..., \tau+m \end{pmatrix}.$$

From the definition of C it follows that $C\binom{1, ..., \tau+m}{k_1, ..., k_{\tau+m}}$ is equal to zero unless $1, ..., \tau$ are all included among the indices $\{k_i\}_{i=1}^{\tau+m}$ and equal to $B\binom{1, ..., m}{k_{\tau+1}-\tau, ..., k_{\tau+m}-\tau}$ if $k_i = i$, $i = 1, ..., \tau$. Therefore

$$\det \Delta_{j_1, ..., j_l} = \sigma \sum_{1 \leqslant i_1 < \cdots < i_m \leqslant \tau} \det K\begin{pmatrix} t_1, ..., t_\tau, i_1 - 1, ..., i_m - 1 \\ j_1' - 1, ..., j_{k-l}' - 1, y_1, ..., y_s \end{pmatrix} \times B\begin{pmatrix} 1, ..., m \\ i_1, ..., i_m \end{pmatrix}.$$

Since (2.3) and

$$(-1)^{j_1 + \cdots + j_l} A\begin{pmatrix} 1, ..., l \\ j_1, ..., j_l \end{pmatrix} = \tilde{A}\begin{pmatrix} 1, ..., l \\ j_1, ..., j_l \end{pmatrix}$$

we have $\sigma_{\tilde{A}} \sigma_B \sigma \det \Delta \geqslant 0$. The proof is complete.

3. Proof of Theorems

The proof of Theorem 2 may proceed as in [2] by use of Theorem 1. So we only give the proof of Theorem 1.

Proof of Theorem 1. Let

$$P_0(t) = \sum_{i=0}^{r-1} a_{0i} t^i + \frac{1}{r!}\left[t^r + 2 \sum_{j=1}^{N} (-1)^j (t - \xi_{0,j})_+^r\right] \tag{3.1}$$

be an arbitrary perfect spline in $\mathscr{P}_{r,N}(A, B; v_1, ..., v_n)$ with n distinct zeros $\{x_{0,i}\}_{i=1}^n \subset (0, 1)$. Since P_0 has no other zero than $\bar{X}_0 = \{(x_{0,i}, v_i)\}_{i=1}^n$ in $(0, 1)$, it holds that

$$(-1)^{\sigma + \sigma_i} P_0(t) \geqslant 0, \qquad t \in (x_{0,i-1}, x_{0,i}), \qquad i = 1, ..., n+1,$$

where $\sigma \in \{-1, 1\}$ fixed, $x_{0,0} = 0$ and $x_{0,n+1} = 1$.

In what follows we have to distinguish the cases as we do in proving Lemma 3. Let α_i and β_i be defined as in Lemma 3 for P_0 (we note that $\alpha_0 = l$, $\beta_0 = m$). We assume first that $\alpha_0 - \alpha_1 = \beta_0 - \beta_1 = 1$. By the proof of Lemma 3, there exist $\{y_{0,i}\}_{i=1}^{n+1}$ any of which is a simple zero of P_0' such that

$$0 < y_{0,1} < x_{0,1} < y_{0,2} < \cdots < x_{0,n} < y_{0,n+1} < 1.$$

Put

$$e_{0,j} = |P_0(y_{0,j})|, \qquad j = 1, ..., n+1,$$

$$e_j(s) = e_{0,j} + s(e_j - e_{0,j}), \qquad j = 1, ..., n+1,\ s \in [0, 1].$$

For $s \in [0, 1]$, we shall construct a function $P(s, \cdot) \in \mathscr{P}_{r,N}(A, B; \nu_1, ..., \nu_n)$, with parameters $a_i(s)$, $x_i(s)$, $y_i(s)$, $\xi_i(s)$, and $R(s)$ such that

$$R(s)\, P(s, y_i(s)) = (-1)^{\sigma + \sigma_i} e_i(s), \qquad i = 1, ..., n+1. \tag{3.2}$$

To this end, we consider the system of equations:

$$\begin{aligned}
P^{(j)}(s, t)|_{t = x_i(s)} &= 0 && i = 1, ..., n,\ j = 0, ..., \nu_i - 1,\\
R(s)\, P(s, y_i(s)) &= (-1)^{\sigma + \sigma_i} e_i(s), && i = 1, ..., n+1,\\
P'(s, t)|_{t = y_i(s)} &= 0, && i = 1, ..., n+1,\\
A_i P(s, \cdot) &= 0, && i = 1, ..., l,\\
B_i P(s, \cdot) &= 0, && i = 1, ..., m,
\end{aligned} \tag{3.3}$$

where $P^{(j)}(s, t) = (\partial^j / \partial t^j)\, P(s, t)$. It is obvious that $(P_0, R_0 = 1)$ satisfies (3.3).

For the ease of computating Jacobian of (3.3) we reorder (3.3) as follows:

$$\begin{aligned}
P^{(\nu_i - 1)}(s, t)|_{t = x_i(s)} &= 0, && i = 1, ..., n,\\
P'(s, t)_{t = y_i(s)} &= 0, && i = 1, ..., n+1,\\
R(s)\, P(s, y_1(s)) &= (-1)^{\sigma + \sigma_1} e_1(s), &&\\
P^{(j)}(s, t)|_{t = x_i(s)} &= 0,\ j = 0, ..., \nu_i - 2, && i = 1, ..., n,\\
R(s)\, P(s, y_{i+1}(s)) &= (-1)^{\sigma + \sigma_{i+1}} e_{i+1}(s), &&\\
A_i P(s, \cdot) &= 0, && i = 1, ..., l,\\
B_i P(s, \cdot) &= 0, && i = 1, ..., m.
\end{aligned} \tag{3.3'}$$

Denote by $\Delta(s)$ the Jacobi matrix of (3.3′) with respect to $\{x_i\}_{i=1}^{n+1}$, $\{y_i\}_{i=1}^{n+1}$, R, $\{a_i\}_{i=0}^{r-1}$ and $\{\xi_i\}_{i=1}^{N}$. Then (cf. [2])

$$\det \Delta(s) = \det J(s) \prod_{i=1}^{n} P^{(\nu_i)}(s, x_i(s)) \prod_{j=1}^{n+1} P''(s, y_i(s))$$

(see (3.10) below for the explanation that each $P^{(\nu_i)}$ is well defined), where $J(s)$ is a $(N+r+1)\times(N+r+1)$ matrix with the first $N+r+1-l-m$ rows

$$\begin{array}{cccccccc} R & a_0 & \cdots & a_{r-1} & \xi_1 & & \cdots & \xi_N \end{array}$$

$$\begin{pmatrix} P(s,y_1) & Ru_0(y_1) & \cdots & Ru_{r-1}(y_1) & \frac{2R(-1)^{1+1}}{(r-1)!}(y_1-\xi_1)_+^{r-1} & \cdots & \frac{2R(-1)^{1+N}}{(r-1)!}(y_1-\xi_N)_+^{r-1} \\ 0 & u_0(x_1) & \cdots & u_{r-1}(x_1) & \frac{2(-1)^{1+1}}{(r-1)!}(x_1-\xi_1)_+^{r-1} & \cdots & \frac{2(-1)^{1+N}}{(r-1)!}(x_1-\xi_N)_+^{r-1} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & u_0^{(\nu_1-2)}(x_1) & \cdots & u_{r-1}^{(\nu_1-2)}(x_1) & \frac{2(-1)^{1+1}}{(r+1-\nu_1)}(x_1-\xi_1)_+^{r-\nu_1+1} & \cdots & \frac{2(-1)^{1+N}}{(r-\nu_1+1)!}(x_1-\xi_N)_+^{r-\nu_1+1} \\ P(s,y_2) & Ru_0(y_2) & \cdots & Ru_{r-1}(y_2) & \frac{2R(-1)^{1+1}}{(r-1)!}(y_2-\xi_1)_+^{r-1} & \cdots & \frac{2R(-1)^{1+N}}{(r-1)!}(y_2-\xi_N)_+^{r-1} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & u_0^{(\nu_n-2)}(x_1) & \cdots & u_{r-1}^{(\nu_n-2)}(x_1) & \frac{2(-1)^{1+1}}{(r+1-\nu_n)!}(x_1-\xi_1)_+^{r-\nu_n+1} & \cdots & \frac{2(-1)^{1+N}}{(r-\nu_n+1)!}(x_1-\xi_N)_+^{r-\nu_n+1} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ P(s,y_{n+1}) & Ru_0(y_{n+1}) & \cdots & Ru_{r-1}(y_{n+1}) & \frac{2R(-1)^{1+1}}{(r-1)!}(y_{n+1}-\xi_1)_+^{r-1} & \cdots & \frac{2R(-1)^{1+N}}{(r-1)!}(y_{n+1}-\xi_N)_+^{r-1} \end{pmatrix}$$

and the last $l+m$ rows

$$\begin{pmatrix} 0_{l\times 1} & (a_{ij}j!)_{l\times r} & 0_{l\times N} \\ 0_{m\times 1} & (B_i u_j)_{m\times r} & \left(\frac{2(-1)^{j+1}}{(r-1)!} B_i K(\cdot, \xi_j)\right)_{m\times N} \end{pmatrix}.$$

Expanding det $J(s)$ based on the first column, we get

$$\det J(s) = \sum_{j=1}^{n+1} (-1)^{\sigma_j} P(s, y_j) \det J_i(s), \tag{3.4}$$

where $J_i(s)$ is the submatrix of $J(s)$ eliminated 1 column and σ_j row.

We use $\{z_{0,1}\leqslant z_{0,2}\leqslant \cdots \leqslant z_{0,p}\}$ $(p=N+r-l-m+1)$ to denote the sequence

$$\{y_{0,1}, (x_{0,1}, v_1-1), y_{0,2}, ..., (x_{0,n}, v_n-1), y_{0,n+1}\},$$

which is the set of zeros of $P_0'\in\mathscr{P}_{r-1,N}$ counting multiplicities. It follows from Lemma 3 and (2.2) that

$$z_{0,j-(l-1)}<\xi_{0,j}<z_{0,j+(r-1)-(l-1)} \tag{3.5}$$

whenever the subscripts are meaningful.

Given any i, put $\{z_{0,j}^{(i)}\}_{j=1}^{p-1}=\{z_{0,j}\}_{j=1}^{p}/\{y_i\}$ arranged in increasing order. Then (3.5) yields

$$z_{0,j-l}^{(i)}<\xi_{0,j}<z_{0,j+r-l}^{(i)}. \tag{3.6}$$

On the other hand we see that $J_i(s)$ has the form of matrix given as in Lemma 4 only different in some rows and columns to some non-zero constants, respectively, independent of i. It follows from Lemma 4, (3.1), and (3.4) that

$$|\det J(0)|=\sum_{j=1}^{n+1}|P_0(y_{0,j})|\ |\det J_i(0)|. \tag{3.7}$$

By the proof of Lemma 4 we have for some positive number γ that

$$|\det J_i(0)|=\gamma \sum_{\substack{1\leqslant i_1<\cdots<i_l\leqslant r\\ 1\leqslant j_1<\cdots<j_m\leqslant r}} \left|\tilde{A}\begin{pmatrix}1, ..., l\\ i_1, ..., i_l\end{pmatrix}\prod_{j=1}^{l}(i_j-1)!\right.$$
$$\left.\times B\begin{pmatrix}1, ..., m\\ j_1, ..., j_m\end{pmatrix}\det K\begin{pmatrix}z_{0,1}^{(i)}, \cdot, z_{0,p-1}^{(i)}, j_1-1, ..., j_m-1\\ i_1'-1, i_{r-l}'-1, \xi_{0,1}, ..., \xi_{0,N}\end{pmatrix}\right|.$$

If $m<N$, then it follows from (3.6) and [7, Theorem 1′] that for any $\{i_k\}_{k=1}^{l}$ and $\{j_k\}_{k=1}^{m}$,

$$\det K\begin{pmatrix}z_{0,1}^{(i)}, ..., z_{0,p-1}^{(i)}, j_1-1, ..., j_m-1\\ i_1'-1, ..., i_{r-l}'-1, \xi_{0,1}, ..., \xi_{0,N}\end{pmatrix}>0. \tag{3.9}$$

Hence, (3.7), rank $A=l$ and rank $B=m$ give $\det J_i(0)\neq 0$.

If $m=N$, the interlacing conditions (3.6) cannot guarantee that (3.9) hold for all $\{i_k\}_{k=1}^{l}$ and $\{j_k\}_{k=1}^{m}$. However, we also have $\det J_i(0)\neq 0$ in this case. In fact, in view of Lemma 4 and [13, Lemma 10], there exists a unique function (i.e., zero element) $s\in\Pi_r(A, B)$ which satisfies

$$s^{(j)}(x_{0,i})=0, \qquad i=1, ..., n, \qquad j=0, ..., v_i-1, \tag{3.10}$$

where

$$\Pi_r = \left\{ s \mid s = \sum_{i=0}^{r-1} a_i t^i + \sum_{i=r}^{r+N-1} a_i (t - \xi_{0,i})_+^{r-1} \right\}.$$

(We have to say a few words about (3.10). By the method of proof of Theorem C of [3] we can conclude that any $\xi_{0,i}$ cannot be a zero of P_0 with multiplicity r, $i = 1, ..., N$. Since $\Pi_r \subset C^{r-2}[0, 1] \cap C^r\{[0, 1]/\{\xi_{0,i}\}_{i=1}^N\}$, all $s^{(j)}(x_{0,i})$ in (3.10) are well defined).

Appealing to [7 Theorem 1′] once again we get two sets of indices $\{i_k^*\}_{k=1}^l$ and $\{j_k^*\}_{k=1}^m$ such that

$$A\begin{pmatrix} 1, ..., l \\ i_1^*, ..., i_l^* \end{pmatrix} B\begin{pmatrix} 1, ..., m \\ j_1^*, ..., j_m^* \end{pmatrix} \neq 0, \tag{3.11}$$

and

$$j_k^* \leqslant i_{k+r+N-l-m}^{*\prime}. \tag{3.12}$$

Therefore, it follows from (3.6), (3.12), and [7, Theorem 1′] that (3.9) holds for $\{i_k\}_{k=1}^l = \{i_k^*\}_{k=1}^l$ and $\{j_k\}_{k=1}^m = \{j_k^*\}_{k=1}^m$. Noting (3.11) we get $\det J_i(0) \neq 0$.

Since (3.7) and other factors in $\det \Delta(0)$ are not zero, we have $\det \Delta(0) \neq 0$. Now we can continue the proof as in Theorem 3.1 in [2] and get the unique $P(\cdot) := P(1, \cdot) \in \mathscr{P}_{r,N}(A, B; v_1, ..., v_n)$ satisfying the theorem. The detail is omitted and referred to [2].

If $\alpha_0 - \alpha_1 = 1$ and $\beta_0 - \beta_1 = 0$, then P_0' has a zero $y_{0,1} \in (0, x_{0,1})$ and not zero in $(x_{0,n}, 1)$. Instead of (3.3) we consider the following system of equations:

$$\begin{aligned}
P^{(j)}(s, t)|_{t = x_i(s)} &= 0, && i = 1, ..., n,\ j = 0, ..., v_i - 1, \\
R(s)\, P(s, y_i(s)) &= (-1)^{\sigma + \sigma_i} e_i(s), && i = 1, ..., n+1, \\
P'(s, t)_{t = y_i(s)} &= 0, && i = 1, ..., n, \\
A_i P(s, \cdot) &= 0, && i = 1, ..., l, \\
B_i P(s, \cdot) &= 0, && i = 1, ..., m,
\end{aligned}$$

where $y_{n+1}(s) = 1$.

As before, we can get a unique $P \in \mathscr{P}_{r,N}(A, B; v_1, ..., v_n)$ satisfying the theorem. The other cases of α_i and β_i may be treated similarly. So Theorem 1 is proved.

Remark 1. Since perfect splines and monosplines may be treated in a unified way (see [4]), Theorems 1 and 2 hold for monosplines. This is related to optimal quadrature formula.

Remark 2. It is not difficult to generalize the results of this paper to the case where separated boundary conditions are replaced by mixed boundary conditions, which were studied, e.g., in [8].

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